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NON-NEGATIVE PERTURBATIONS OF NON-NEGATIVE SELF-ADJOINT OPERATORS

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Abstract. Let A be a non-negative self-adjoint operator in a Hilbert space $\mathcal{H}$ and A_0 be some densely defined closed restriction of A_0, $A_0 \subseteq A \neq A_0$. It is of interest to know whether A is the unique non-negative self-adjoint extensions of A_0 in $\mathcal{H}$. We give a natural criterion that this is the case and if it fails, we describe all non-negative extensions of A_0. The obtained results are applied to investigation of non-negative singular point perturbations of the Laplace and poly-harmonic operators in $\mathbb{L}_2(\mathbf{R}_n)$.

1. Introduction

In this paper we deal with a non-negative self-adjoint operator A in a Hilbert space$\mathcal{H}$, some densely defined not essentially self-adjoint restriction A_0 of A and again with self-adjoint extensions of A_0 in $\mathcal{H}$, which following [1] we call here *singular perturbations* of A. For quick getting onto the matter of main problem let us compare the point perturbations of self-adjoint Laplace operators $-\Delta$ in three and two dimensions acting in $\mathbb{L}_2(\mathbf{R}_3)$ and $\mathbb{L}_2(\mathbf{R}_2)$, respectively, that is let us consider the restriction $-\Delta^0$ of $-\Delta$ onto the Sobolev subspaces $\mathbb{H}_2^2(\mathbf{R}_i \setminus \{0\})$, $i = 3, 2$ and self-adjoint extensions $-\Delta_\alpha$, $\alpha \in \mathbf{R}$ of $-\Delta^0$ in $\mathbb{L}_2(\mathbf{R}_i)$ with domains

(1.1)
$$\mathcal{D}_\alpha^{(3)} := \left\{ f : \, f \in \mathbb{H}_2^2(\mathbf{R}_3), \, \lim_{|\mathbf{x}| \downarrow 0} \left[\tfrac{d}{d|\mathbf{x}|} \left(|\mathbf{x}| f(\mathbf{x}) \right) - \alpha |\mathbf{x}| f(\mathbf{x}) \right] = 0 \right\},$$
$$\mathcal{D}_\alpha^{(2)} := \left\{ f : \, f \in \mathbb{H}_2^2(\mathbf{R}_2), \, \lim_{|\mathbf{x}| \downarrow 0} \left[\left(\tfrac{2\pi\alpha}{\ln|\mathbf{x}|} + 1 \right) f(\mathbf{x}) - \lim_{|\mathbf{x}'| \downarrow 0} \tfrac{\ln|\mathbf{x}|}{\ln|\mathbf{x}'|} f(\mathbf{x}') \right] = 0 \right\}.$$

The self-adjoint operators $-\Delta_\alpha$ are just mentioned above singular perturbations of $-\Delta$. Resolvents $(-\Delta_\alpha - z)^{-1}$, $z \in \rho(-\Delta_\alpha)$, of operators $-\Delta_\alpha$ act in the corresponding spaces $\mathbb{L}_2$ as integral operators with kernels (Green functions) [1]:

$$G_{\alpha,z}^3(\mathbf{x}, \mathbf{x}') = \begin{cases} G_z^{(0)}(\mathbf{x}, \mathbf{x}') + (\alpha - i\sqrt{z}/4\pi)^{-1} G_z^{(0)}(\mathbf{x}, 0) G_z^{(0)}(0, \mathbf{x}'), \\ G_z^{(0)}(\mathbf{x}, \mathbf{x}') = \frac{\exp i\sqrt{z}|\mathbf{x}-\mathbf{x}'|}{4\pi|\mathbf{x}-\mathbf{x}'|} \text{ (three dimension)}; \end{cases} \tag{1.2}$$

(1.3)
$$G_{\alpha,z}^2(\mathbf{x}, \mathbf{x}') = \begin{cases} G_z^{(0)}(\mathbf{x}, \mathbf{x}') + 2\pi(2\pi\alpha - \psi(1) + \ln\sqrt{z}/2i)^{-1} G_z^{(0)}(\mathbf{x}, 0) G_z^{(0)}(0, \mathbf{x}'), \\ G_z^{(0)}(\mathbf{x}, \mathbf{x}') = (\frac{i}{4}) H_0^{(1)}(i\sqrt{z}|\mathbf{x} - \mathbf{x}'|) \text{ (two dimension)}. \end{cases}$$

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By (1.2) the Green function $G_{\alpha,z}(\mathbf{x},\mathbf{x}')$ of self-adjoint operator $-\Delta_\alpha$ in $\mathbb{L}_2(\mathbf{R}_3))$ is holomorphic on the half-axis $(-\infty,0)$ for $\alpha \geq 0$ and has on this half-axis a simple pole for $\alpha < 0$. Hence in the case of three dimensions self-adjoint extensions $-\Delta_\alpha$ are non-negative for all $(\alpha \geq 0)$ and non-positive for $\alpha < 0$.

Contrary to this by (1.3) in the case of two dimensions for any $\alpha \in \mathbf{R}$ the Green function $G_{\alpha,z}$ has a simple pole on the half-axis $(-\infty,0)/$. Hence all singular perturbations $-\Delta_\alpha$ of the two-dimensional Laplace operators have one negative eigenvalue. In other words the standardly defined Laplace operator $-\Delta$ is the unique non-negative self-adjoint extension in $\mathbb{L}_2(\mathbf{R}_2)$ of the symmetric operator $-\Delta^0$ in $\mathbb{L}_2(\mathbf{R}_2)$.[1]

In this note we try to reveal the underlying cause of such discrepancy. Remind that each densely defined non-negative symmetric operator has at least one non-negative canonical self-adjoint extension (Friedrichs extension). In more general setting we try to understand here why in some cases the non-negative extension appears to be unique. Actually this questions is embedded into the framework of the general extension theory for semi-bounded symmetric operators developed in the famous paper of M.G. Krein [2]. Naturally, there is a criterium of uniqueness of non-negative extension in [2]. In the next Section using only approaches of [2] we find another form of this criterium directly facilitated to investigation of singular perturbations and for cases where conditions of these criterium fail describe all non-negative singular perturbations of a given non-negative self-adjoint operator A associated with some its densely defined non-self-adjoint restriction A_0. In fact we give here a parametrization of the operator interval $[A_\mu, A_M]$ of all canonical non-negative self-adjoint extensions of a given densely defined non-negative operator. The third Section illustrates obtained results by the example of singular perturbations of Laplace and poly-harmonic operators in $\mathbb{L}_2(\mathbf{R}_n)$.

Note that very close results were obtained recently in somewhat different way in [3], where in terms of this note were described singular perturbations of the Friedrichs extension of a given densely defined non-negative operator and also with illustration by the example of singular perturbations of the Laplace operator in $\mathbb{L}_2(\mathbf{R}_3)$.

2. Uniqueness criterium and parametrization of non-negative singular perturbations

Let A be a non-negative self-adjoint operator acting in the Hilbert space $\mathcal{H}$ and A_0 be a densely defined closed operator, which is a restriction of A onto a subset $\mathcal{D}(A_0)$ of the domain $\mathcal{D}(A)$ of A. Let us consider the subspaces $\mathcal{M} := (I + A_0\mathcal{D}(A_0)$ and $\mathcal{N} := \mathcal{H} \ominus \mathcal{M}$. We will assume that

$$1)\ \mathcal{M} \neq \mathcal{H}, \quad 2)\ \mathcal{N} \cap \mathcal{D}(A) = \{0\}. \tag{2.1}$$

We call all self-adjoint extensions of A_0 in $\mathcal{H}$ other than the given A singular perturbations of A. It is of interest to know whether there are non-negative operators among singular perturbations of A. In this section we try to find a convenient criterium that such singular perturbations of A does not exist. In other words we look for a criterium that A is one and only non-negative operator among all self-adjoint extensions of A_0. Following the approach developed in the renowned paper of M.G.

[1]The attention of author to this phenomenon was drawn by Sergey Gredeskul.

Krein [2] let us consider the operator from $K_0 : \mathcal{M} \to \mathcal{H}$ defined by relations

$$f = (I + A_0)\, x, \quad K_0 f = \mathcal{A}_0 x, \quad x \in \mathcal{D}(A_0). \tag{2.2}$$

It is easy to see that K_0 is a non-negative contraction:

$$(K_0 f, f) \geq 0, \quad \|K_0 f\|^2 \leq \|f\|^2, \quad f \in \mathcal{M}. \tag{2.3}$$

Let A_1 be any non-negative self-adjoint extension of A_0 in $\mathcal{H}$. Then $K_1 := A_1\,(A_1 + I)^{-1}$ is a non-negative operator, which is a contactive extension of K_0 from the domain $\mathcal{M}$ onto the whole $\mathcal{H}$, $K_1 f = K_0 f$, $f \in \mathcal{M}$.

From the other hand for any contractive extension K_1 from $\mathcal{M}$ onto $\mathcal{H}$ such that the unity is not its eigenvalue the non-negative self-adjoint operator $A_1 = K_1\,(I - K_1)^{-1}$ is a self-adjoint extension of A_0 in $\mathcal{H}$. Therefore A_0 *has unique non-negative self-adjoint extension in* $\mathcal{H}$ *if and only if* K_0 *admits only one non-negative contractive extension onto the whole* $\mathcal{H}$, *no eigenvalue of which* $= 1$, *that is* $K = A(I + A)^{-1}$. So the uniqueness of A as non-negative extension of A_0 is equivalent to uniqueness of K_0 as non-negative contractive extension of K_0.

From now on we will denote by $\mathbf{G}$ the set consisting of A and all its singular perturbations and by $\mathbf{C}$ the set of non-negative contractions obtained from $\mathbf{G}$ by transformation $A_1 \to A_1\,(A_1 + I)^{-1}$, $A_1 \in \mathbf{G}$. Let us denote by $P_{\mathcal{M}}$ the orthogonal projector onto $\mathcal{M}$ in $\mathcal{H}$ and let $P_{\mathcal{N}} = I - P_{\mathcal{M}}$. With respect to representation of $\mathcal{H}$ as the orthogonal sum $\mathcal{M} \oplus \mathcal{N}$ we can represent each operator from $\mathbf{C}$ as 2×2 block operator matrix

$$K_X = \begin{pmatrix} T & \Gamma^* \\ \Gamma & X \end{pmatrix} \tag{2.4}$$

Here

$$T = P_{\mathcal{M}} K_0|_{\mathcal{M}}, \quad \Gamma = P_{\mathcal{M}} K_0|_{\mathcal{M}}.$$

and X is some non-negative contraction in $\mathcal{N}$, which distinguishes different elements from $\mathbf{C}$. Since each $K_X \in \mathbf{C}$ is non-negative and contractive then

$$T \geq 0; \quad T^2 + \Gamma^*\Gamma \leq I \tag{2.5}$$

Note further that

$K_X \in \mathbf{C}$ *is equivalent to*

$$K_X + \varepsilon I \geq 0; \quad (1 + \varepsilon)I - K_X \geq 0 \tag{2.6}$$

for any $\varepsilon > 0$.

The block matrix representation of K_X and the Schur -Frobenius factorization formula transform (2.6) into the following block matrix inequalities:

$$\begin{pmatrix} I & 0 \\ \Gamma(T+\varepsilon)^{-1} & I \end{pmatrix} \begin{pmatrix} T+\varepsilon & 0 \\ 0 & X + \varepsilon - \Gamma(T+\varepsilon)^{-1}\Gamma^* \end{pmatrix} \times \begin{pmatrix} I & (T+\varepsilon)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0, \tag{2.7}$$

$$\begin{pmatrix} I & 0 \\ -\Gamma(I+\varepsilon-T)^{-1} & I \end{pmatrix} \begin{pmatrix} 1+\varepsilon-T & 0 \\ 0 & 1+\varepsilon-X-\Gamma(1+\varepsilon-T)^{-1}\Gamma^* \end{pmatrix} \times \begin{pmatrix} I & -(1+\varepsilon-T)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0. \tag{2.8}$$

By our assumptions $T \geq 0$ and $I - T \geq 0$. Therefore block matrix inequalities (2.7) and (2.8) are reduced to

$$(2.9) \qquad \left\{ \begin{array}{c} X + \varepsilon I - \Gamma(T+\varepsilon I)^{-1}\Gamma^* \geq 0, \\ (1+\varepsilon)I - X - \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* \geq 0, \ \varepsilon > 0. \end{array} \right.$$

Observe that operator functions of ε in the left hand sides of inequalities (2.9) are monotone. Setting

$$Y := X - \lim_{\varepsilon \downarrow 0} \Gamma(T+\varepsilon I)^{-1}\Gamma^*$$

we conclude from (2.9) that $K_X \in \mathbf{C}$ if and only if

$$(2.10) \qquad 0 \leq Y \leq I - \lim_{\varepsilon \downarrow 0} \left\{ \Gamma(T+\varepsilon I)^{-1}\Gamma^* + \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* \right\}.$$

Hence the equality

$$(2.11) \qquad I - \lim_{\varepsilon \downarrow 0} \left\{ \Gamma(T+\varepsilon I)^{-1}\Gamma^* + \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* \right\} = 0$$

is the criterium that there are no contractive non-negative extension of K_0 in $\mathcal{H}$ other than K.

Let us express now (2.10) and 2.11) in terms of given K and A. To this end we use the following proposition.

Proposition 2.1. *Let L be a bounded invertible operator in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ given as $2 \times @$ block operator matrix,*

$$L = \begin{pmatrix} R & U \\ V & S \end{pmatrix}$$

, where R and S are invertible operators in $\mathcal{M}$ and $\mathcal{N}$, respectively, and U, V act between $\mathcal{M}$ and $\mathcal{N}$. If R is invertible operator in $\mathcal{M}$, then

$$(2.12) \qquad \begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} = L^{-1} - L^{-1}P_{\mathcal{N}}\Lambda^{-1}P_{\mathcal{N}}L^{-1}, \ \Lambda = P_{\mathcal{N}}L^{-1}|_{\mathcal{N}}.$$

Setting

$$(2.13) \qquad \Lambda_{1,\varepsilon} = P_{\mathcal{N}}(K+\varepsilon I)^{-1}|_{\mathcal{N}} \quad \Lambda_{2,\varepsilon} = P_{\mathcal{N}}[(1+\varepsilon)I - K]^{-1}|_{\mathcal{N}}$$

and applying (2.12) with $L = K + \varepsilon I$ and

$$\begin{array}{cc} R = T + \varepsilon I & U = \Gamma^* = P_{\mathcal{M}}K|_{\mathcal{N}} = P_{\mathcal{M}}[K+\varepsilon I]|_{\mathcal{N}} \\ V = \Gamma = P_{\mathcal{N}}K|_{\mathcal{M}} = P_{\mathcal{N}}[K+\varepsilon I]|_{\mathcal{M}} & S = P_{\mathcal{N}}K|_{\mathcal{N}} + \varepsilon I \end{array}$$

yields

$$\Gamma(T+\varepsilon I)^{-1}\Gamma^* = P_{\mathcal{N}}K|_{\mathcal{N}} + \varepsilon I - \Lambda_{1,\varepsilon}^{-1}.$$

In the same fashion we get

$$\Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* = P_{\mathcal{N}}[I-K]|_{\mathcal{N}} + \varepsilon I - \Lambda_{2,\varepsilon}^{-1}.$$

Hence

$$(2.14) \qquad I - \lim_{\varepsilon \downarrow 0} \left(\Gamma(T+\varepsilon I)^{-1}\Gamma^* + \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* \right) = \lim_{\varepsilon \downarrow 0} \Lambda_{1,\varepsilon}^{-1} + \lim_{\varepsilon \downarrow 0} \Lambda_{2,\varepsilon}^{-1}.$$

Combining (2.10), (2.11) and (2.14) results in the following theorem.

Theorem 2.2. *Let K be a non-negative contraction in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, K_0 is the restriction of K onto the subspace $\mathcal{M}(= \mathcal{M} \oplus \{0\})$ and*

$$G_1 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[K + \varepsilon I]|_{\mathcal{N}}\right)^{-1} \quad G_2 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[I - K + \varepsilon I]|_{\mathcal{N}}\right)^{-1}$$

Then the set $\mathbf{C}$ of all non-negative contractive extensions K_X of K_0 in $\mathcal{H}$ is described by expression

$$K_X = \begin{pmatrix} P_{\mathcal{M}}K|_{\mathcal{M}} & P_{\mathcal{M}}K|_{\mathcal{N}} \\ P_{\mathcal{M}}K|_{\mathcal{N}} & X \end{pmatrix}, \tag{2.15}$$

where X runs the set of all non-negative contractions in $\mathcal{N}$ satisfying inequalities

$$P_{\mathcal{N}}K|_{\mathcal{N}} - G_1 \leq X \leq P_{\mathcal{N}}K|_{\mathcal{N}} + G_2. \tag{2.16}$$

In particular, K is the unique non-negative contractive extension of K_0 if and only if $G_1 = G_2 = 0$.

Remark 2.3. The set $\mathbf{C}$ of non-negative contractive of K_0 contains the minimal extension K_{X_μ} with $X_\mu = P_{\mathcal{N}}K|_{\mathcal{N}} - G_1$ in (2.15) and the maximal extension K_{X_M} with $X_M = P_{\mathcal{N}}K|_{\mathcal{N}} + G_2$ in (2.15. If $G_1 = 0\,(G_2 = 0)$, then K is the minimal (maximal) element of $\mathbf{C}$.

Theorem 2.2 can be formulated in terms of non-negative self-adjoint operator A and its non-negative singular perturbations.

Theorem 2.4. *Let A be a non-negative self-adjoint operator in the Hilbert space $\mathcal{H}$, A_0 is a densely defined closed symmetric operator, which is a restriction of A onto a linear subset $\mathcal{D}(A_0) \subset \mathcal{D}(A)$ such that $\mathcal{N} = (I + A)\mathcal{D}(A_0) \neq \{0\}$ and let*

$$G_1 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[I + A][A + \varepsilon I]|_{\mathcal{N}}\right)^{-1} \quad G_2 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[I + A][I + \varepsilon A]|_{\mathcal{N}}\right)^{-1}$$

Then the set of all non-negative singular perturbations A_Y of A is described by the formula

$$\begin{cases} f = g - Y(I + A)g, \\ A_Y f = Ag + Y(I + A)g, \end{cases} \tag{2.17}$$

where $g \in \mathcal{D}(A)$ and Y runs the set of non-negative contractions in $\mathcal{N}$ satisfying inequalities

$$-G_1 \leq Y \leq G_2. \tag{2.18}$$

A has no singular non-negative perturbations if and only if $G_1 = G_2 = 0$.

Remark 2.5. The set of all non-negative singular perturbations of A contains the minimal perturbation A_μ with and the maximal perturbation A_M such that any non-negative perturbation A_X satisfies inequalities

$$(I + A_M)^{-1} \leq A_Y \leq (I + A_\mu)^{-1}.$$

The corresponding values of parameters Y in Theorem 2.4 are

$$\begin{matrix} Y_\mu = -G_1 \\ Y_M = G_2 \end{matrix} \tag{2.19}$$

If $G_1 = 0\,(G_2 = 0)$, then the minimal (maximal) perturbation coincides with A.

By simple calculation we get from (2.17) the following version of the M.G. Krein resolvent formula.

Proposition 2.6. *The set of resolvents of all non-negative singular perturbations A_Y of A is described by the M.G. Krein formula*

$$(A_Y - zI)^{-1} = (A - zI)^{-1}$$
$$-(1+z)(A+I)(A-zI)^{-1}Y\left[I + (1+z)P_{\mathcal{N}}(A+I)(A-zI)^{-1}Y\right]^{-1} \times \tag{2.20}$$
$$P_{\mathcal{N}}(A+I)(A-zI)^{-1},$$

where Y runs contractions in $\mathcal{N}$ satisfying inequalities $-G_1 \leq Y \leq G_2$.

3. Application to some differential operators

Let us consider the multiplication operator A in $\mathbb{L}_2(\mathbf{R}_n)$ by the continuous function $\varphi(k)$, $k^2 = k_1^2 + ... + k_n^2$, such that $\varphi(k) > 0$ almost everywhere and

$$\int_0^\infty \frac{1}{(1+\varphi(k))^2} k^{n-1} dk < \infty. \tag{3.1}$$

A is a non-negative self-adjoint operator,

$$\mathcal{D}(A) = \left\{ f : \int_{\mathbb{R}_n} |1+\varphi(k)|^2 |f(\mathbf{k}|^2 d\mathbf{k} < \infty,\ f \in \mathbb{L}_2(\mathbf{R}_n) \right\}.$$

In the sequel $\hat{\delta}$ stands for the unbounded linear functional in $\mathbb{L}_2(\mathbf{R}_n)$, formally defined as follows:

$$\hat{\delta}(f) = \int_{\mathbb{R}_n} f(\mathbf{k}) d\mathbf{k}.$$

Note that the domain of $\hat{\delta}$ contains $\mathcal{D}(A)$. Let us denote by A_0 the restriction of A onto linear set

$$\mathcal{D}_0(A) := \left\{ f : f \in \mathcal{D}(A),\ \hat{\delta}(f) = 0. \right\} \tag{3.2}$$

The closure of $A_0 \neq A$ and

$$\mathcal{N} = (\mathbb{L}_2(\mathbf{R}_n) \ominus (I+A)\mathcal{D}_0(A)) = \left\{ \xi \cdot \frac{1}{1+\varphi(k)},\ \xi \in \mathbf{C} \right\}.$$

Applying Theorem 2.4 yields

Proposition 3.1. *A is the unique non-negative self-adjoint extension of A_0 that is A has no non-negative singular perturbations if and only if*

$$\int_0^\infty \tfrac{1}{\varphi(k)(1+\varphi(k))} k^{n-1} dk = \infty \quad \text{and} \quad \int_0^\infty \tfrac{1}{(1+\varphi(k))} k^{n-1} dk = \infty. \tag{3.3}$$

Put $\varphi(k) = k^2$ and let $n = 2$. Then the both integrals in Proposition 3.1 are divergent. Hence the restriction A_0 of the operator A of multiplication by k^2 in $\mathbb{L}_2(\mathbf{R}_2)$ onto the linear set (3.2 has unique non-negative self-adjoint extension in $\mathbb{L}_2$. Note that the multiplication operator by k^2 in $\mathbb{L}_2(\mathbf{R}_n$ is isomorphic to the self-adjoint Laplace operator $-\Delta$ in $\mathbb{L}_2(\mathbf{R}_n$ and its concerned here restriction A_0 is isomorphic to the restriction $-\Delta$ onto the Sobolev subspace $\mathbb{H}_2^2(\mathbf{R}_n \setminus \{0\})$. As follows, the self-adjoint Laplace operator in $\mathbb{L}_2(\mathbf{R}_2$ has no non-negative singular perturbations with support at one point of $\mathbf{R}_2$.

However, the non-negative singular perturbations of $-\Delta$ in $\mathbb{L}_2(\mathbf{R}_2$ with support at two or more points do already exist. For example, let us consider there the

restriction A_0 of the multiplication operator by k^2, for which the defect subspace $\mathcal{N}$ is one-dimensional and consists of functions collinear to

$$e_0(\mathbf{k}) = \frac{1-\exp(-i(\mathbf{k}\cdot\mathbf{x}_0))}{1+k^2},\ \mathbf{x}_0 \in \mathbf{R}_2.$$

In this case

$$\begin{gathered} \|e_0\|^2 = \int_{\mathbf{R}_2} \frac{4\sin^2\frac{1}{2}(\mathbf{k}\cdot\mathbf{x}_0)}{(1+k^2)^2}\cdot d\mathbf{k} < \infty, \\ \big((I+A)A^{-1}e_0, e_0\big) = \int_{\mathbf{R}_2} \frac{4\sin^2\frac{1}{2}(\mathbf{k}\cdot\mathbf{x}_0)}{k^2(1+k^2)}\cdot d\mathbf{k} < \infty, \\ ((I+A)e_0, e_0) = \int_{\mathbf{R}_2} \frac{4\sin^2\frac{1}{2}(\mathbf{k}\cdot\mathbf{x}_0)}{1+k^2}\cdot d\mathbf{k} = \infty. \end{gathered}$$

Hence $G_1 = \|e_0\|^2\cdot((I+A)e_0,e_0)^{-1} > 0$, but $G_2 = 0$. As follows, the concerned restriction A_0 of the multiplication operator A by k^2 has non-negative self-adjoint extensions in $\mathbb{L}_2(\mathbf{R}_2$ others then A and A is the maximal element in the set of these extensions. It remains to note that A is isomorphic to the self-adjoint Laplace operator $-\Delta$ in $\mathbb{L}_2(\mathbf{R}_2)$ and A_0 is isomorphic to the restriction of this $-\Delta$ on the subset of function $f(\mathbf{x})$ from $\mathcal{D}(-\Delta)$ satisfying conditions

$$\begin{gathered} \lim_{|\mathbf{x}|\to 0}(\ln|\mathbf{x}|)^{-1}f(\mathbf{x}) - \lim_{|\mathbf{x}-\mathbf{x}_0|\to 0}(\ln|\mathbf{x}-\mathbf{x}_0|)^{-1}f(\mathbf{x}) = 0, \\ \lim_{|\mathbf{x}|\to 0}\left[f(\mathbf{x}) - \ln|\mathbf{x}|\lim_{|\mathbf{x}'|\to 0}(\ln|\mathbf{x}'|)^{-1}f(\mathbf{x}')\right] - \\ \lim_{|\mathbf{x}-\mathbf{x}_0|\to 0}\left[f(\mathbf{x}) - \ln|\mathbf{x}-\mathbf{x}_0|\lim_{|\mathbf{x}'-\mathbf{x}_0|\to 0}(\ln|\mathbf{x}'-\mathbf{x}_0|)^{-1}f(\mathbf{x}')\right] = 0. \end{gathered}$$

Put now as above $\varphi(k) = k^2$ and let $n = 3$. Then the first integral in Proposition 3.1 is convergent while the second one as before divergent. Hence the restriction A_0 of the operator A of multiplication by k^2 in $\mathbb{L}_2(\mathbf{R}_3)$ onto the linear set (3.2 has infinitely many non-negative self-adjoint extension in $\mathbb{L}_2(\mathbf{R}_3)$. As follows, the self-adjoint Laplace operator in $\mathbb{L}_2(\mathbf{R}_3)$ has infinitely many non-negative singular perturbations with support at one point of $\mathbf{R}_3$ and the standardly defined Laplace the maximal element in the set of this perturbation.

As the next example we consider the multiplication operator A by k^{2l} in $\mathbb{L}_2(\mathbf{R}_n)$ assuming that $4l \le n+1$. A is isomorphic to the polyharmonic operator $(-\Delta)^l$ in $\mathbb{L}_2(\mathbf{R}_n)$. Let us consider the restriction A_0 of A with the domain (3.2) that is non-negative symmetric operator which is isomorphic to the restriction of the polyharmonic operator $(-\Delta)^l$ onto the Sobolev subspace $\mathbb{H}^2_{2l}(\mathbf{R}_n\setminus\{0\})$. Applying Theorem 2.4 and Proposition 3.1 results in the following proposition.

Proposition 3.2. *If $n < 2l$ then there are infinitely many non-negative singular perturbations of $(-\Delta)^l$ associated with the one-point symmetric restriction A_0 and $(-\Delta)^l$ is the minimal element in the set of the non-negative extensions of A_0 in $\mathbb{H}^2_{2l}(\mathbf{R}_n\setminus\{0\})$.*

If $n = 2l$ then $(-\Delta)^l$ has no such perturbations in $\mathbb{H}^2_{2l}(\mathbf{R}_n\setminus\{0\})$.

If $n > 2l$ then there is the infinite set of non-negative singular perturbations of $(-\Delta)^l$ associated with A_0 and for those as non-negative extensions of A_0 in the set of the in $\mathbb{H}^2_{2l}(\mathbf{R}_n\setminus\{0\})$ the operator $(-\Delta)^l$ is the maximal element.

References

1. Albeverio S., Gesztesy F., Høegh-Krohn R. and Holden H. *Solvable models in quantum mechanics*, Springer Verlag (1988), 1 - 566
2. Krein M.G., *Theory of self-adjoint extensions of semibounded Hermitian operators and its applications. I*, Mathematical reports (Mat. Sbornik (in Russian)), v. **20**(**62**), 3 (1947)
3. Arlinskii Yu. and Tsekanovskii E. *The von Neumann Problem for Nonnegative Symmetric Operators*, Intgr. equ.oper.theory **51** (2005), 319 -356

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